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LETTER TO THE EDITOR

q-boson realization of quadratic algebra $\mathscr{A}_1$ and its representations

Hong-Chen Fu†‡ and Mo-Lin Ge†

† Theoretical Physics Divison, Nankai Institute of Mathematics, Tianjin 300071, People's Republic of China

‡ Department of Physics, Northeast Normal University, Changchun 130024, People's Republic of China

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Abstract. The non-generic central elements of the quadratic algebra $\mathscr{A}_1$ associated with the quantum group $GL(2)_q$ are found in the case where q is a root of unity. A q-boson realization of $\mathscr{A}_1$ is constructed. In terms of the q-boson realization the representations of $\mathscr{A}_1$ on the q-Fock space are studied in both generic and non-generic cases and the cyclic representation is obtained in the non-generic case.

Reflection equations and their related quadratic algebras were introduced in [1] as an equation describing factoring scattering on a half-line. Recently they have found different applications, to the quantum current algebras [2], and to the integrable modules with non-periodic boundary conditions [3, 4]. Kulish *et al* studied the properties of the quadratic algebras [5] (including some representations) and constructed the constant solutions of the reflection equations [6].

The q-boson realization theory is a powerful tool for studying the representations of quantum algebras [7], quantum superalgebras [8], and quantum matrix-element algebras of the quantum groups [9]. We naturally expect to apply this method to the study of the representations of the quadratic algebras. This letter is devoted to the q-boson realization of quadratic algebras $\mathscr{A}_1$ (in Kulish's notation) associated with $GL(2)_q$ and its representations. The q-Fock representations both in generic and non-generic cases and the cyclic representations in non-generic cases are all considered.

Throughout this letter the term 'generic' means that the deformation parameter q is not a root of unity, and 'non-generic' means that q is the primitive pth root of unity, i.e. $q^p = 1$, and $p \geqslant 3$ is an odd positive integer. We denote by Z^+ the set of all non-negative integers and by C the complex number field. We also use the abbreviations $C^\times = C\backslash\{0\}$ and $[x] = (q^x - q^{-x})/(q - q^{-1})$ for an operator x or a complex number x.

Quadratic algebra $\mathscr{A}(R)$ was specified from the reflection equation without spectral parameters

$$RK^1R^{t_1}K^2 = K^2R^{t_1}K^1R \tag{1}$$

where K is a square matrix and $K^1 = K\otimes\mathrm{id}$, $K^2 = \mathrm{id}\otimes K^3$. Quadratic algebras are generated by the non-commuting matrix elements k_{ij} of K. This algebra is closely related to the quantum group $A(R)$ generated by the matrix elements t_{ij} of T satisfying

$$R\overset{1}{T}\overset{2}{T} = \overset{2}{T}\overset{1}{T}R. \tag{2}$$

It is an A(R)-comodule-algebra, i.e. there exists an algebra homomorphism φ: $\mathscr{A}(R)\rightarrow A(R)\otimes\mathscr{A}(R)$ such that

$$(\Delta\otimes\mathrm{id})\circ\varphi=(\mathrm{id}\otimes\varphi)\circ\varphi \qquad (\varepsilon\otimes\mathrm{id})\circ\varphi=\mathrm{id} \tag{3}$$

where Δ and ε are the comultiplication and the co-unit of A(R) respectively. In fact φ is explicitly defined by

$$\varphi(K)=TKT' \qquad (\varphi(K))_{ij}=\sum_{m,n} t_{im}t_{jn}k_{mn} \tag{4}$$

provided $[t_{ij}, k_{mn}]=0$. This property implies that, if K is a solution of equation ( ), then $\varphi(K)$ is also a solution.

In this letter we only study an explicit example $\mathscr{A}_1$ associated with quantum group $\mathrm{GL}(2)_q$. In this case

$$R=\begin{bmatrix} q & & & \\ & 1 & & \\ & \omega & 1 & \\ & & & q \end{bmatrix} \tag{5}$$

where $\omega=q-q^{-1}$. Letting

$$K=\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \tag{6}$$

we get the defining relations of $\mathscr{A}_1$

$$\begin{aligned} &[\alpha,\beta]=\omega\alpha\gamma \qquad \alpha\gamma=q^2\gamma\alpha \qquad [\alpha,\delta]=\omega(q\beta+\gamma)\gamma \\ &[\beta,\gamma]=0 \qquad [\beta,\delta]=\omega\gamma\delta \qquad \gamma\delta=q^2\delta\gamma. \end{aligned} \tag{7}$$

This algebra has two central elements

$$C_1=\beta-q\gamma \qquad C_2=\alpha\delta-q^2\beta\gamma. \tag{8}$$

For the non-generic case we can also prove the following proposition.

Proposition 1. If $q^p=1$, then α^p, γ^p, δ^p are all the central elements of $\mathscr{A}_1$.

This proposition can be proved from the defining relations (7) and the following equations

$$\begin{aligned} &[\alpha,\delta^m]=\omega[m]\delta^{m-1}(q^m\beta+(q^{3m-1}-q^m\omega)\gamma)\gamma \\ &[\beta,\delta^m]=q^{-(m-1)}[m]\omega\gamma\delta^m \\ &[\alpha^m,\delta]=\omega\alpha^{m-1}[m](q^{-m+2}\beta+q^{-3m+3}\gamma)\gamma \\ &[\alpha^m,\beta]=q^{-(m-1)}[m]\omega\alpha^m\gamma. \end{aligned} \tag{9}$$

We note that, if p is even, then $\gamma^{p/2}$ is the central element. It is also worth noting that β^p is not the non-generic central element. For example, when $p=3$, we find that

$$[\alpha,\beta^3]=3\omega\beta^2\alpha\gamma+3\omega^2\beta\alpha\gamma^2+\omega^3\alpha\gamma^3\neq 0. \tag{10}$$

This is explained in greater detail as follows. In quantum algebras, there are two kinds of *Cartan generators*: one kind is H_i, having the property $[H_i, X_j^{\pm}]=\pm A_{ij}X_j^{\pm}$; another kind is $k_i\equiv q^{H_i}$ satisfying the relations $k_iX_j^{\pm}=q^{\pm A_{ij}}X_j^{\pm}k_i$. An important difference between the two kinds of *Cartan generators* is that, in the non-generic case, k_i^p are the central elements, while H_i^p are not. From the defining relations (7) one finds that β is H_i-like and γ is k_i-like; therefore, γ^p are the central elements while β^p are not. This fact can also be seen from the q-boson realization constructed later.

We now construct the q-boson realization of $\mathscr{A}_1$. Define

$$\begin{aligned} &\delta = b^+ \qquad \beta = \lambda + \omega\mu[N]q^{N+1} \\ &\gamma = \mu q^{2N} \qquad \alpha = \omega\mu(q^{N+1}\lambda + (q^{3N+2} - q^{N+1}\omega)\mu)b \end{aligned} \tag{11}$$

where the q-boson operators b^+, b, $q^{\pm N}$ satisfy the following well known relations [7]

$$bb^+ - q^{\mp 1}b^+b = q^{\pm N} \qquad q^N b^+ q^{-N} = qb^+ \qquad q^N b q^{-N} = q^{-1}b. \tag{12}$$

One can verify that these operators indeed satisfy the defining relations (7) of the quadratic algebra $\mathscr{A}_1$, and therefore equations (11) give a q-boson realization of $\mathscr{A}_1$.

One can also verify that, in the q-boson realization (11), the central elements C_1, C_2 take the constants

$$C_1 = \lambda - q\mu \qquad C_2 = \mu(-\lambda + \omega\mu). \tag{13}$$

We turn to the representation of the $\mathscr{A}_1$ on the q-Fock space $\mathscr{F}$ spanned by

$$\{|m\rangle = (b^+)^m|0\rangle | b|0\rangle = 0, q^{\pm N}|0\rangle = |0\rangle, m \in Z^+\}. \tag{14}$$

From the representations of q-boson algebra on $\mathscr{F}$

$$\begin{aligned} &b^+|m\rangle = |m+1\rangle \\ &b|m\rangle = [m]|m-1\rangle \\ &q^{\pm N}|m\rangle = q^{\pm m}|m\rangle \end{aligned} \tag{15}$$

we obtain the q-Fock representation of $\mathscr{A}_1$

$$\begin{aligned} &\delta|m\rangle = |m+1\rangle \\ &\beta|m\rangle = (\lambda + q^{m+1}[m]\omega\mu)|m\rangle \\ &\gamma|m\rangle = \mu q^{2m}|m\rangle \\ &\alpha|m\rangle = \omega[m]\mu(q^m\lambda + (q^{3m-1} - q^m\omega)\mu)|m-1\rangle. \end{aligned} \tag{16}$$

Let us analyse this representation in different cases.

Case 1: $\mu = 0$. In this case the representation becomes

$$\begin{aligned} &\delta|m\rangle = |m+1\rangle \\ &\beta|m\rangle = \lambda|m\rangle \\ &\gamma|m\rangle = \alpha|m\rangle = 0. \end{aligned} \tag{17}$$

It is easy to see that there exists the following invariant subspace chain

$$\mathscr{F} \equiv V(0) \supset V(1) \supset V(2) \supset \ldots \tag{18}$$

where the invariant subspace $V(M)$, $M \in Z^+$, is spanned by

$$V(M)\colon \{|m\rangle | m \geqslant M\}. \tag{19}$$

It is not difficult to probe that for the subspace $V(M+1)$ of $V(M)$ there exists no invariant complementary space. Thus the representation on every $V(M)$ is an infinite-dimensional indecomposable representation.

On the quotient space $V(M, K) = V(M)/V(M+K)$, $K = 1, 2, \ldots$, one can obtain the finite-dimensional representations, which are one-dimensional irreducible representations if $K = 1$, and K-dimensional indecomposable ones if $K \geqslant 2$. The one-dimensional representation reads

$$\alpha = \gamma = \delta = 0 \qquad \beta = \lambda. \tag{20}$$

Case 2: $\mu \neq 0$ *and* $\lambda \neq (\omega - q^{2\Lambda-1})\mu$ *for any* $\Lambda \in Z^+$. In this case it is easy to prove that equation (16) defines an infinite-dimensional irreducible representation in the generic case. If $q^p = 1$, there exists the following invariant subspace chain

$$\mathscr{F} \equiv W(0) \supset W(p) \supset W(2p) \supset \ldots \tag{21}$$

where the invariant subspaces $W(Rp)$, $R \in Z^+$, are spanned by

$$W(Rp): \{|m\rangle | m \geqslant Rp\} \qquad R \in Z^+ \tag{22}$$

for which no invariant complementary space exists. Therefore the representations on $\mathscr{F}$ and on $W(Rp)$ are all the infinite-dimensional indecomposable representations.

From the chain (21) we can also construct the finite-dimensional representations on the quotient spaces $W(R, S) \equiv W(R)/W(R+S)$, $S = 1, 2, \ldots$, which are spanned by

$$\begin{aligned} &W(R, S): \{\overline{|m\rangle} \equiv |m\rangle \bmod W(R+S) | Rp \leqslant m \leqslant (R+S)p - 1\} \\ &\dim W(R, S) = Sp. \end{aligned} \tag{23}$$

These finite-dimensional representations are indecomposable in the case $S \geqslant 2$ and irreducible in the case $S = 1$.

Case 3: $\mu \neq 0$ *and* $\lambda = (\omega - q^{3\Lambda-1})\mu$ *for given* $\Lambda \in Z^+$. We first consider the generic case. In this case there exists an invariant subspace $F(\Lambda)$ spanned by

$$F(\Delta): \{|m\rangle | m \geqslant \Lambda\} \tag{24}$$

for which no invariant complementary space exists. Therefore the representation on $\mathscr{F}$ is indecomposable.

On the quotient space $\mathscr{F}/F(\Delta)$ with basis

$$\begin{aligned} &\{\overline{|m\rangle} \equiv |m\rangle \bmod F(\Lambda) | 0 \leqslant m \leqslant \Lambda - 1\} \\ &\dim F(\Lambda) = \Lambda \end{aligned} \tag{25}$$

we obtain a Λ-dimensional irreducible representation. In particular, when $\Lambda = 1$, we obtain the well known one-dimensional representation (up to the constant μ)

$$\alpha = \delta = 0 \qquad \beta = 1 \qquad \gamma = -q. \tag{26}$$

Next we discuss the non-generic case. In this case there exists the invariant subspace chain (21) and the invariant subspace $F(\Lambda)$. If $\Lambda = Tp$ $(T \in Z^+)$, then $F(\Lambda)$ is just one of the chain, thus the explanation is the same as case 2 with $q^p = 1$. If $\Lambda \neq Tp$, and letting $Rp < \Lambda < (R+1)p$, we get the following invariant chain

$$\begin{aligned} &\mathscr{F} \equiv W(0) \supset W(p) \supset W(2p) \supset \ldots \supset \\ &W(Rp) \supset F(\Lambda) \supset W(Rp+p) \supset \ldots \end{aligned} \tag{27}$$

on each of which we have an infinite-dimensional indecomposable representation.

In this case we can obtain a new type of finite-dimensional representations on the quotient spaces $F(P, \Lambda) \equiv W(Pp)/F(\Lambda)$, $P \leqslant R$, with basis

$$\begin{aligned} &\{\overline{|m\rangle} \equiv |m\rangle \bmod F(\Lambda) | Pp \leqslant m \leqslant \Lambda - 1\} \\ &\dim F(P, \Lambda) = \Lambda - Pp \end{aligned} \tag{28}$$

which is irreducible if $P = R$ and indecomposable if $P < R$.

The q-boson realization method can also be used to study the cyclic representations [10]. Now we study the cyclic representation of $\mathscr{A}_1$. On the p-dimensional linear space V_p with basis $\{v_k | k = 0, 1, \ldots, p-1\}$ the q-boson algebra has the cyclic representation

$$
\begin{aligned}
& b^+ v_k = v_{k+1} && k \neq p-1 \\
& b^+ v_{p-1} = \xi v_0 && \xi \in C^\times \\
& b v_k = [k+\eta] v_{k-1} && k \neq 0,\ \eta \text{ is generic} \\
& b v_0 = \xi^{-1} [\eta] v_{p-1} && \\
& q^{\pm N} v_k = q^{\pm(k+\eta)} v_k. &&
\end{aligned} \tag{29}
$$

Then, by making use of the q-boson realization (11) of $\mathscr{A}_1$, we obtain the cyclic representation of $\mathscr{A}_1$

$$
\begin{aligned}
& \delta v_k = v_{k+1} && k \neq p-1 \\
& \delta v_{p-1} = \xi v_0 && \xi \in C^\times \\
& \beta v_k = (\lambda + \omega\mu[k+\eta] q^{k+\eta+1}) v_k && \eta \text{ is generic} \\
& \gamma v_k = \mu q^{2(k+\eta)} v_k && \\
& \alpha v_k = \omega\mu[k+\eta](q^{k+\eta}\lambda + (q^{3(k+\eta)-1} - q^{k+\eta}\omega)\mu) v_{k-1} && k \neq 0 \\
& \alpha v_0 = \xi^{-1}\omega\mu[\eta](q^{\eta}\lambda + (q^{3\eta-1} - q^{\eta}\omega)\mu) v_{p-1}. &&
\end{aligned} \tag{30}
$$

In this representation the non-generic central elements take the values

$$
\begin{aligned}
& \delta^p = \xi \qquad \gamma^p = \mu^p \\
& \alpha^p = \omega^p \mu^p \xi^{-1} \Pi_{k=0}^{p-1} [\eta + k] \Pi_{k=0}^{p-1} (q^{\eta+k}\lambda + (q^{3(\eta+k)-1} - q^{\eta+k}\omega)\mu).
\end{aligned} \tag{31}
$$

We would like to point out that in the case with $\mu = 0$ or in the case with $\mu \neq 0$ but $\lambda = (\omega - q^{2(\Lambda+\eta)-1})\mu$ for a $\Lambda \in \{0, 1, \ldots, p-1\}$ the representation (30) is only the semi-cyclic representation. It is obvious that in both cases we always have $\alpha^p = 0$.

So far we have studied the q-boson realization and the representation of quadratic algebra $\mathscr{A}_1$. The key point is the construction of the q-boson realization. In fact, by making use of the comodule property of the quadratic algebras we can also construct q-boson realizations, which are different from the q-boson realizations presented, in this letter, for the case $\mathscr{A}_1$. We will present this approach in a separate paper.

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